CSC 611 – Algorithms and Advanced Data Strcutures
Exam #3, Fall 2024

First/Given Name: ____________________________

Last/Family Name: ____________________________

This exam contains 5 pages (including this cover page) and 3 questions.

- Clearly identify your answer for each problem, and try to organize your work in a reasonably coherent way, in the space provided. If you decided to use the back of a paper, note this clearly so the instructor can find your answer. You can simplify and shortify answers by combining terms, reducing fractions, etc, to an extent that it still shows you understand what are you doing.
- It might be a good idea to draw a box around your final answer.
- Partial credit will be given for incorrect answers that show a partial understanding of the relevant concepts. Therefore, it is a good idea to show your work to convince your instructor that you understand the material. Irrelevant and meaningless answers will not receive partial credit.
- No electronic devices, including calculators, are allowed.
- You have 30 minutes to complete this exam.
- Each student is allowed to use a cheat sheet of size $4.5'' \times 5.75''$, which is equivalent to one-fourth of a standard letter-sized paper. The cheat sheet can be used on both sides. Only hand-written cheat sheets are allowed, and each student is required to write their name on their cheat sheet. The cheat sheet must be submitted along with the exam upon completion.

Question	Points	Score
1	2.00	
2	2.00	
3	2.00	
Total:	6.00	

I acknowledge that it is the responsibility of every student at Missouri State University to adhere to the university's policies on Student Academic Integrity. I confirm that I have neither given nor received any unauthorized assistance during this exam.

Signature: ____________________________

1. (2.00 points) Use the substitution method to prove that if

$$T(n) = 4T(n/3) + n,$$

then $T(n) = O(n^{\log_3 4})$.

Solution: We first assume that $T(n) = cn^{\log_3 4}$.

Given the induction assumption that $T(m) \le cm^{\log_3 4}$ for all $m < n$, we need to prove that $T(n) \le cn^{\log_3 4}$.

$$\begin{aligned} T(n) &= 4T(n/3) + n \\ &\le 4\left(c(\frac{n}{3})^{\log_3 4}\right) + n \quad \text{[Ind. Assum. has been applied]} \\ &= 4c\frac{n^{\log_3 4}}{3^{\log_3 4}} + n \\ &= 4c\frac{n^{\log_3 4}}{4} + n \\ &= cn^{\log_3 4} + n \end{aligned}$$

Note that this is a dead end because we cannot provide any value for c to make the inequality $cn^{\log_3 4} + n \le cn^{\log_3 4}$ hold. Thus, we let $T(n) = cn^{\log_3 4} - an$, and provide value for a along the proof.

Given the induction assumption that $T(m) \le cm^{\log_3 4} - am$ for all $m < n$, we need to prove that $T(n) \le cn^{\log_3 4} - an$

$$\begin{aligned} T(n) &= 4T(n/3) + n \\ &\le 4\left(c(\frac{n}{3})^{\log_3 4} - a\frac{n}{3}\right) + n \quad \text{[Ind. Assum. has been applied]} \\ &= 4c\frac{n^{\log_3 4}}{3^{\log_3 4}} - \frac{4}{3}an + n \\ &= 4c\frac{n^{\log_3 4}}{4} - \frac{4}{3}an + n \\ &= cn^{\log_3 4} - an - \frac{1}{3}an + n \\ &\le cn^{\log_3 4} - an \qquad [a \ge 3] \end{aligned}$$

Note that the range for a is obtained by letting $-\frac{1}{3}an + n \le 0$.

 Points earned for this question: ______

2. (2.00 points) Use the Master theorem to solve the following recurrences.

   (a) $T(n) = 4T(n/2) + n^3$

   Solution: We use the simple version of the Master theorem because $T(n) = aT(n/b) + n^d$ where $a = 4$, $b = 2$, and $d = 3$. Because $a < b^d$, we use the third case of the Master theorem, and thus, $T(n) = \Theta(n^d) = \Theta(n^3)$.

   (b) $T(n) = 9T(n/3) + n^2 \lg n$

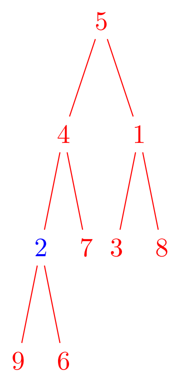
   Solution: We use the general version of the Master theorem. Because $n^2 \lg n = \Theta(n^{\log_3 9} \lg^k n)$ for $k = 1$, the second case of the Master theorem applies, and thus, $T(n) = \Theta(n^{\log_3 9} \lg^{k+1} n) = \Theta(n^2 \lg^2 n)$.

Points earned for this question: ______

3. (2.00 points) Illustrate the operations of BUILD-MAX-HEAP on the following array:

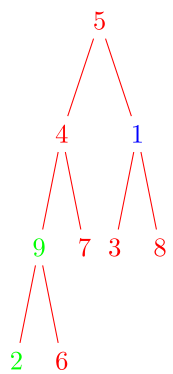
$$A = \langle 5 \quad 4 \quad 1 \quad 2 \quad 7 \quad 3 \quad 8 \quad 9 \quad 6 \rangle$$

Draw the binary tree represented by the initial array, and then, list the comparisons and swaps, in order, and draw the final binary tree.

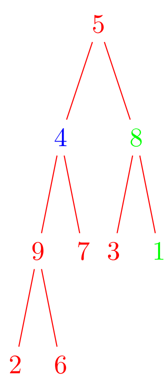


Solution: Initial Tree:

- $i = 9/2 = 4$.
  - Compare 2 with 9: $9 > 2$
  - Compare 9 with 6: $9 > 6$



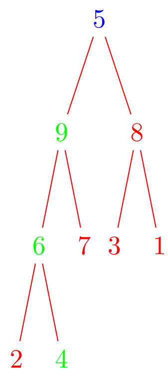
  - Swap 9 and 2
- $i = 3$
  - Compare 1 with 3: $3 > 1$
  - Compare 3 with 8: $8 > 3$



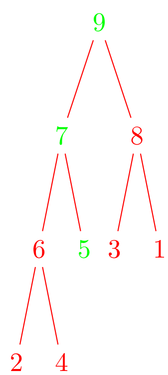
  - Swap 8 with 1

 Points earned for this question: ______

- $i = 2$
  - Compare 4 with 9: $9 > 4$
  - Compare 9 with 7: $9 > 7$
  - Swap 4 with 9
  - Compare 4 with 2: $4 > 2$
  - Compare 4 with 6: $6 > 2$



  - Swap 6 and 4
- $i = 1$
  - Compare 5 with 9: $9 > 5$
  - Compare 9 with 8: $9 > 8$
  - Swap 5 with 9
  - Compare 5 with 6: $6 > 5$
  - Compare 6 with 7: $7 > 6$



  - Swap 5 and 7

 Points earned for this question: ______