CSC 611 – Algorithms and Advanced Data Strcutures
Exam #2, Fall 2024

First/Given Name: ____________________________

Last/Family Name: ____________________________

This exam contains 4 pages (including this cover page) and 3 questions.

- Clearly identify your answer for each problem, and try to organize your work in a reasonably coherent way, in the space provided. If you decided to use the back of a paper, note this clearly so the instructor can find your answer. You can simplify and shortify answers by combining terms, reducing fractions, etc, to an extent that it still shows you understand what are you doing.
- It might be a good idea to draw a box around your final answer.
- Partial credit will be given for incorrect answers that show a partial understanding of the relevant concepts. Therefore, it is a good idea to show your work to convince your instructor that you understand the material. Irrelevant and meaningless answers will not receive partial credit.
- No electronic devices, including calculators, are allowed.
- You have 40 minutes to complete this exam.
- Each student is allowed to use a cheat sheet of size $4.5'' \times 5.75''$, which is equivalent to one-fourth of a standard letter-sized paper. The cheat sheet can be used on both sides. Only hand-written cheat sheets are allowed, and each student is required to write their name on their cheat sheet. The cheat sheet must be submitted along with the exam upon completion.

Question	Points	Score
1	2.00	
2	2.00	
3	2.00	
Total:	6.00	

I acknowledge that it is the responsibility of every student at Missouri State University to adhere to the university's policies on Student Academic Integrity. I confirm that I have neither given nor received any unauthorized assistance during this exam.

Signature: ____________________________

1. (2.00 points) Arrange the following functions from the left to right in ascending order by asymptotic growth rate. If two or more functions have the same rate of growth (that is, when they are in the Θ of one another), put them in the same class by arranging them vertically in the list or putting them in a set.

$$0.1n^3 - n \qquad \lg(n^n) \qquad 2^{2\lg n} \qquad 2^n \qquad n^2 \qquad (0.1n)^n \qquad n\lg n \qquad n\lg\lg 3^n$$

Solution:

$$\{n\lg n, \lg(n^n), n\lg\lg 3^n\} \qquad \{n^2, 2^{2\lg n}\} \qquad 0.1n^3 - n \qquad 2^n \qquad (0.1n)^n$$

We used the following simplifications.

$$\lg(n^n) = n\lg n \qquad [\log_b a^c = c\log_b a]$$

$$\begin{aligned} 2^{2\lg n} &= 2^{\lg n^2} && [c\log_b a = \log_b a^c] \\ &= n^2 && [b^{\log_b x} = x] \end{aligned}$$

$$\begin{aligned} n\lg\lg 3^n &= n\lg(n\lg 3) && [\log_b a^c = c\log_b a] \\ &= n(\lg n + \lg\lg 3) && [\log_b(xy) = \log_b x + \log_b y] \\ &= n\lg n + n\lg\lg 3 \\ &= \Theta(n\lg n) && [\lg\lg 3 \text{ is a constant}] \end{aligned}$$

 Points earned for this question: ______

2. (2.00 points) Find a simple formula (with no summations) for the following summation.

$$\sum_{k=1}^{n} \frac{2}{(k+1)(k+2)}$$

Solution: Let

$$\frac{2}{(k+1)(k+2)} = \frac{A}{k+1} + \frac{B}{k+2}.$$

We need to compute A and B. To do so, we multiply both sides by $(k+1)(k+2)$, resulting in

$$2 = A(k+2) + B(k+1).$$

By choosing $k = -2$, we have

$$\begin{aligned} 2 &= A(-2+2) + B(-2+1) \\ &= -B. \end{aligned}$$

Thus, $B = -2$.

And, by choosing $k = -1$, we have

$$\begin{aligned} 2 &= A(-1+2) + B(-1+1) \\ &= A. \end{aligned}$$

Thus, $A = 2$.

Therefore,

$$\begin{aligned} \sum_{k=1}^{n} \frac{2}{(k+1)(k+2)} &= \sum_{k=1}^{n} \left(\frac{2}{k+1} - \frac{2}{k+2} \right) \\ &= \left(\frac{2}{2} - \frac{2}{3} \right) + \left(\frac{2}{3} - \frac{2}{4} \right) + \cdots \left(\frac{2}{n+1} - \frac{2}{n+2} \right) \\ &= \frac{2}{2} - \frac{2}{n+2} = 1 - \frac{2}{n+2} = \frac{n}{n+2}. \end{aligned}$$

Points earned for this question: ______

3. In this question, you are asked to prove asymptotic upper and lower bounds for a summation. Please use any method you wish, but show that you understand what you are doing.

   (a) (1.00 points) First, prove that

$$\sum_{k=1}^{n} \sqrt{k} = O(n\sqrt{n}).$$

Solution: Consider that for each $1 \leq k \leq n$, $\sqrt{k} \leq \sqrt{n}$. Therefore,

$$\begin{aligned}\sum_{k=1}^{n} \sqrt{k} &\leq \sum_{k=1}^{n} \sqrt{n} \\ &= n\sqrt{n}.\end{aligned}$$

Therefore, if we chose $n_0 = 1$ and $c_1 = 1$, then for all $n \geq n_0$,

$$\sum_{k=1}^{n} \sqrt{k} \leq c_1 n\sqrt{n}.$$

This proves that

$$\sum_{k=1}^{n} \sqrt{k} = O(n\sqrt{n}).$$

   (b) (1.00 points) Then, prove that

$$\sum_{k=1}^{n} \sqrt{k} = \Omega(n\sqrt{n}).$$

Solution: Observe that

$$\begin{aligned}\sum_{k=1}^{n} \sqrt{k} &\geq \sum_{k=\lceil \frac{n}{2} \rceil}^{n} \sqrt{k} \\ &\geq \sum_{k=\lceil \frac{n}{2} \rceil}^{n} \sqrt{\frac{n}{2}} \\ &\geq \frac{n}{2} \cdot \sqrt{\frac{n}{2}} \\ &= \frac{1}{2\sqrt{2}} n\sqrt{n}.\end{aligned}$$

Accordingly, if we chose $n_0 = 1$ and $c_2 = \frac{1}{2\sqrt{2}}$, then for all $n \geq n_0$,

$$\sum_{k=1}^{n} \sqrt{k} \geq c_2 n\sqrt{n}.$$

This proves that

$$\sum_{k=1}^{n} \sqrt{k} = \Omega(n\sqrt{n}).$$

 Points earned for this question: ______